EN5101 Digital Control Systems

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1. State Space Model / State transition
2. State Feedback Control
3. State Observer / Observer Design
4. Z-Transforms
5. Pulse Response, Stability, distortions
6. Digital Redesign
7. Lyapunov Stability
8. Kalman Filer

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State Space Model
State Transition

From ODE to State Space Model

consider the system described by the second order ODE

$$\frac{d^2y(t)}{dt^2} + a_2\frac{dy(t)}{dt} + a_0 y(t) = b_0 \underbrace{u(t)}_{} \quad \text{——(1)}$$

↑control input
exogeneous input

$$\xleftarrow{\frac{d}{dt}} x_2(t) \xleftarrow{\frac{d}{dt}} x_1(t)$$

$$\left.\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -a_0 x_1(t) - a_1 x_2(t) + b_0 u(t) \end{aligned}\right\} (2)$$

There are two state variables $x_1(t) = y(t)$, and $x_2(t) = dy(t)/dt$. And the state space model is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a_0 x_1 - a_1 x_2 + b_0 u$$

In matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ b_0\end{bmatrix}u \qquad (3)$$

$$\dot{x} = \underbrace{A}_{\substack{\text{system matrix}\\ \text{Homogeneous part}\\ \text{Autoregressive part}}} x + \underbrace{b}_{\substack{\text{exogeneous part}\\ \text{drive matrix}}} u$$

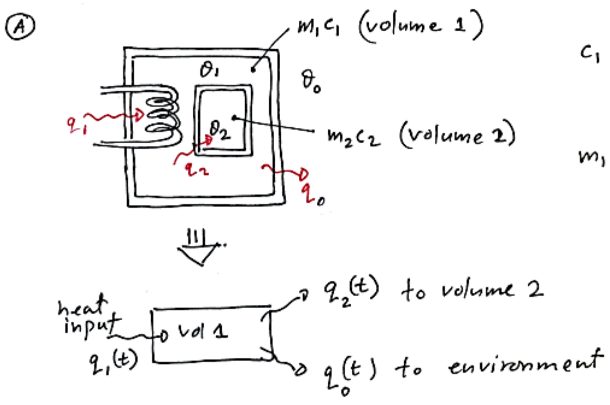
$$y = C'X$$

$$C' = [1\ \ 0], \qquad [1\ \ 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

C': output matrix (vector)

Example: Temp Control System

Ⓐ



c_1, c_2 : Specific heat capacity of volume 1 and 2.

$m_1\ m_2$: masses of volume 1 and 2.

Ⓑ modeling : Heat transfer dynamics

- change of accumulated heat of a volume causes its temperature to change
- Heat flow from one volume to another is proportional to their temperature difference.

for vol 1 :

$$m_1 c_1 \dot{\theta}_1(t) = \underbrace{q_1(t)}_{in} - \underbrace{q_0(t) - q_2(t)}_{out}$$

$$m_1 c_1 \dot{\theta}_1(t) = q_1(t) - K_{10}(\theta_1(t) - \theta_0(t)) - K_{12}(\theta_1(t) - \theta_2(t))$$

for vol 2 :

$$m_2 c_2 \dot{\theta}_2(t) = q_2(t)$$

$$= K_{12}(\theta_1(t) - \theta_2(t))$$

Ⓒ state space model : Arrange (1) and (2) as follows.

(1) ⇒ $m_1 c_1 \dot{\theta}_1(t) = -(K_{10} + K_{12})\theta_1(t) + K_{12}\theta_2(t) + K_{10}\theta_0(t) + q_1(t)$

$$\dot{\theta}_1(t) = -\frac{(K_{10}+K_{12})}{m_1 c_1}\theta_1(t) + \frac{K_{12}}{m_1 c_1}\theta_2(t) + \frac{K_{10}}{m_1 c_1}\theta_0(t) + \frac{1}{m_1 c_1}q_1(t)$$

(2) ⇒ $\dot{\theta}_2(t) = \frac{K_{12}}{m_2 c_2}\theta_1(t) - \frac{K_{12}}{m_2 c_2}\theta_2(t)$

In matrix form

constant ambiant temperature

$$\begin{bmatrix}\dot{\theta}_1(t)\\ \dot{\theta}_2(t)\end{bmatrix} = \begin{bmatrix} \frac{-(k_{10}+k_{12})}{m_1c_1} & \frac{k_{12}}{m_1c_1}\\ \frac{k_{12}}{m_2c_2} & \frac{-k_{12}}{m_2c_2}\end{bmatrix}\begin{bmatrix}\theta_1(t)\\ \theta_2(t)\end{bmatrix} + \begin{bmatrix}\frac{k_{10}}{m_1c_1} & \frac{1}{m_1c_1}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\theta_0\\ q(t)\end{bmatrix}$$

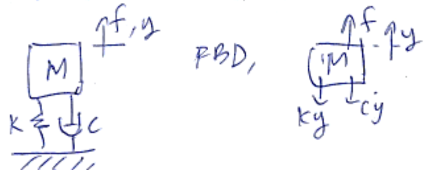
$$\dot{\underline{x}} = A\,\underline{x} + B\,u$$

The output equation is

$$\left.\begin{aligned}\theta_2(t) &= \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\theta_1(t)\\ \theta_2(t)\end{bmatrix}\\ y &= C\,\underline{x}\end{aligned}\right\}\ \text{State space model of the plant}$$

Assignment #1

Derive the state space model of the plant



f: input, exogeneous signal a disturbance.

Answer

State Transition without Control Input

$$\dot{x}(t) = A\,x(t) \quad —(4)$$

(a) Take Laplace

$$s\,X(s) - x(0) = A\,X(s)$$

$$(sI-A)\,X(s) = x(0)$$

$$X(s) = (sI-A)^{-1}x(0)$$

$$x(t) = \underbrace{\mathcal{L}^{-1}\{(sI-A)^{-1}\}}_{\Phi_t}\,x(0) \quad —(5)$$

(b) Assume $x(t) = e^{At}x(0)$ is a solution of (4)

Then $\dot{x}(t) = A\,\underbrace{e^{At}x(0)}$ by differentiation

We have $\dot{x}(t) = A\ x(t)$ which is (4),

Therefore $x(t) = \underbrace{e^{At}}_{\Phi_t}x(0)$ is a solution of (4) $\longrightarrow$ (6)

Therefore $x(t) = \underbrace{e^{At}}_{\Phi_t}x(0)$ is a solution of (4) $\longrightarrow$ (6)

However, (5) and (6) are equivalent expressions for $x(t)$, Thus we have

$$\underline{\underline{\mathcal{L}^{-1}\{(sI-A)^{-1}\} = e^{At}}} = \Phi_t \quad \text{state transition matrix} \quad —(7)$$

It can also be proven that

$$\underline{\underline{e^{At} = \sum_{k=0}^{\infty}\frac{A^k t^k}{k!}}} \quad \text{(computer implementation of matrix exponential)} \quad —(8)$$

Cayley-Hamilton

Example: Obtain the state transition matrix of the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

* Homogeneous state eqn
* no exogeneous input.

The system matrix $A=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$

state transition Matrix $\Phi(t)=\mathcal{L}^{-1}\left[(sI-A)^{-1}\right]$

$$(sI-A)=\begin{pmatrix}s & 0\\ 0 & s\end{pmatrix}-\begin{pmatrix}0 & 1\\ -2 & -3\end{pmatrix}$$

$$=\begin{pmatrix}s & -1\\ 2 & s+3\end{pmatrix}$$

$$(sI-A)^{-1}=\frac{1}{(s+2)(s+1)}\begin{bmatrix}s+3 & 1\\ -2 & s\end{bmatrix}$$

$= s(s+3)--2$

$$=\begin{bmatrix}\frac{(s+3)}{(s+2)(s+1)} & \frac{1}{(s+2)(s+1)}\\ \frac{-2}{(s+2)(s+1)} & \frac{s}{(s+2)(s+1)}\end{bmatrix}$$

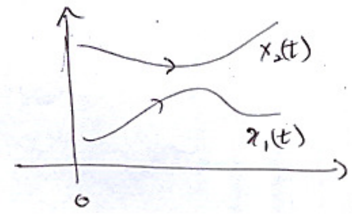
$$\Phi=\mathcal{L}^{-1}\left[(sI-A)^{-1}\right]=\mathcal{L}^{-1}\begin{bmatrix}\frac{2}{(s+1)}-\frac{1}{(s+2)} & \frac{1}{(s+1)}-\frac{1}{(s+2)}\\ \frac{-2}{(s+1)}+\frac{2}{(s+2)} & \frac{-1}{(s+1)}+\frac{2}{(s+2)}\end{bmatrix}$$

$$=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$$

$$\therefore\quad x(t)=\Phi_t\,x(0)$$

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}$$

initial condition



State transition with control input

$$(3)\Rightarrow\ \dot{x}(t)=A\,x(t)+B\,u(t)$$

Take Laplace

$$sX(s)-x(0)=AX(s)+BU(s)$$

$$(sI-A)X(s)=x(0)+BU(s)$$

$$X(s)=(sI-A)^{-1}x(0)+(sI-A)^{-1}BU(s)$$

new term convolution integral

Take inverse Laplace

$$x(t)=\underbrace{\mathcal{L}^{-1}\left[(sI-A)^{-1}\right]}\,x(0)+\mathcal{L}^{-1}\left[(sI-A)^{-1}\right]BU(s)\quad -(9)$$

$$x(t)=\Phi_t\,x(0)+\int_0^t\Phi(t-\tau)\,Bu(\tau)\,d\tau\quad -(10)$$

(we derived this)

convolution integral

Transition from $t=t_0$ to $t=t$ can be determind as follows, at $t=t_0$, from (9)

$$x(t_0)=\Phi_{t_0}x(0)+\int_0^{t_0}\Phi(t_0-\tau)\,Bu(\tau)\,d\tau$$

Therefore, $x(0)=\Phi_{t_0}^{-1}\left[x(t_0)-\int_0^{t_0}\Phi(t_0-\tau)\,Bu(\tau)\,d\tau\right]$ — (11)

Substitute on (10)

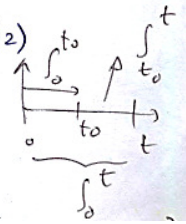
$$x(t)=\overset{e^{At}}{\Phi_t}\,\overset{e^{-At_0}}{\Phi_{t_0}^{-1}}\left[x(t_0)-\int_0^{t_0}\Phi(t_0-\tau)\,Bu(\tau)\,d\tau\right]+\int_0^t\Phi(t-\tau)\,Bu(\tau)\,d\tau$$

$$=\overset{e^{A(t-t_0)}}{\Phi_{t,t_0}}\,x(t_0)-\overset{e^{A(t-t_0)}}{\Phi_{t,t_0}}\int_0^{t_0}\overset{e^{A(t_0-\tau)}}{\Phi(t_0-\tau)}\,Bu(\tau)\,d\tau+\int_0^t\overset{e^{A(t-\tau)}}{\Phi(t-\tau)}\,Bu(\tau)\,d\tau$$

$\Phi(t-\tau)$

$$x(t)=\Phi_{t,t_0}\,x(t_0)+\int_{t_0}^t\Phi(t-\tau)\,Bu(\tau)\,d\tau\quad -(12)$$

State Transition equation $t_0\rightarrow t$



Example

Example : obtain the state transition equation of the system for unit step input

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}}_{A}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \underbrace{\begin{pmatrix}0\\ 1\end{pmatrix}}_{B} u$$

state transition equation is

$$x(t) = \Phi_t\, x(0) + \int_0^t \Phi(t-\tau)\, B\, u(\tau)\, d\tau$$

from previous example $\Phi_t = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t}\end{bmatrix}$

and

$$\int_0^t \Phi(t-\tau)\, B\, \underset{1}{\underset{\parallel}{u(\tau)}}\, d\tau = \int_0^t \left[\underbrace{\begin{matrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)}\end{matrix}}_{\text{meaningless elements}} \quad \underbrace{\begin{matrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -e^{-(t-\tau)} + 2e^{-2(t-\tau)}\end{matrix}}_{\text{meaningful elements}}\right]\begin{pmatrix}0\\ 1\end{pmatrix} d\tau$$

Two meaningful integrations are as follows

$$\int_0^t e^{-(t-\tau)} - e^{-2(t-\tau)}\, d\tau = \left[e^{-(t-\tau)} - \tfrac{1}{2} e^{-2(t-\tau)}\right]_{\tau=0}^{t} =$$

$$= \left[(1 - 1/2) - (e^{-t} - 1/2\, e^{-2t})\right] = 0.5 - e^{-t} + 0.5 e^{-2t}$$

$$\int_0^t -e^{-(t-\tau)} + 2e^{-2(t-\tau)}\, d\tau = \left[-e^{-(t-\tau)} + e^{-2(t-\tau)}\right]_{\tau=0}^{t}$$

$$= \left[(-1+1) - (-e^{-t} + e^{-2t})\right] = e^{-t} - e^{-2t}$$

Therefore, state transition equation is

$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t}\end{bmatrix}\begin{pmatrix} x_1(0) \\ x_2(0)\end{pmatrix} + \begin{bmatrix} 0.5 - e^{-t} + 0.5 e^{-2t} \\ e^{-t} - e^{-2t}\end{bmatrix}$$